

Microtorsion

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The problem of unification of electro-magnetism and gravitation in four dimensions; some new ideas involving torsion. A metric consisting of a combination of symmetric and anti-symmetric parts is postulated and, in the framework of general covariance, used to derive the free-field electro-magnetic stress-energy tensor and the source tensor.

I. INTRODUCTION

H. Weyl, by the use of scale transformations, attempted to unify gravitation and electro-magnetism within the framework of general relativity early this century [1] and in so doing initiated the ‘gauge revolution’ in physics. Like Einstein and many others who followed, Weyl recognised that the two forces which propagate at the speed of light must be intimately connected. Today we understand that there are profound similarities between electromagnetism and gravitation in spite of the obvious differences; both are gauge field theories, both are mediated by massless bosons (if we accept the reality of gravitons) and both manifest as waves in the vacuum in classical (non-quantum) theory. However at present these two forces are described by very different physical principles; in the case of gravitation by a metric theory of general relativity which relates gravity to space-time structure whilst in the classical (and quantum) field theory of electro-magnetism space-time geometry is only in the background. Thus whilst the standard descriptions are not contradictory they are also not cohesive; intuitively we feel that two such similar forces should be based on similar physical principles. Indeed, the general idea that nature, at its most fundamental level of structure, should be simple would seem to require a single set of physical principles to underpin these two forces lest nature be required to ‘reinvent’ itself to create two forces based on two quite different foundations.

Thus few would dispute the need for a unified theory but the dichotomy persists despite nearly a century of work. The discovery of the weak and strong interactions has further complicated the picture; the successful $SU(3)_C \times SU(2)_L \times U(1)$ resisting the incorporation of the (nonrenormalisable) Einstein Lagrangian.

Recently deAndrade and Pereira [2] have pointed out that, in addition to the known result that General Relativity can be recast into an equivalent gauge theory of the translation group due to teleparallel geometry [3] [4] (leading to ‘dual’ descriptions of gravitation - one describing gravitation as propagating space-time curvature and the other ‘teleparallel’ description describing gravitation as propagating torsion), electromagnetism additionally can have such a dual description and that the gauge invariance of the theory is in fact NOT violated by the coupling to torsion. This is in contradistinction to the usual wisdom which precludes torsion coupling to Proca’s equation for $m = 0$ [8] so that theories of torsion in electro-magnetism usually imply photon mass. More will be said about this apparent conflict later - and solutions proposed - but consider the following. If it is possible to have ‘dual’ descriptions of gravitation might it be possible to have ‘dual’ descriptions of electro-magnetism which in some way ‘complement’ gravity theory?

Consider the motivation for this proposition a different way. The Coleman-Mandula (C-M or no-go theorem) is the rock which bars the way for unification. This theorem forbids the (non-trivial) union of compact groups (such as $U(1)$) and non-compact groups (such as the Poincaré group or the Lorentz group). However, operators which interconvert bosons and fermions bypass the theorem; this is the underlying motivation for supersymmetry. This theory however requires a whole menagerie of superpartner particles for which there is currently no empirical evidence. Whilst it is thus assumed that the superpartners are more massive than currently accessible energies the situation is somewhat unsatisfactory. An alternative is highly desirable.

Part of the motivation for this paper is an attempt to avoid the C-M theorem by creating ‘dual’ and complementary descriptions of electromagnetism in a single metric theory. Roughly speaking what is formed is a teleparallel version of electromagnetism with zero non-metricity (curvature=0, torsion \neq 0 for the free-field) but with substantial differences from previous attempts. Chief among these is the attempt to mirror supersymmetry by the creation of a spinorial representation for bosonic torsion. Normally we interpret a metric as defining distances in space-time for an observer. Any component of a metric which defines a $ds^2 = 0$ component does not contribute to such a length; i.e. it does not define a measurement in an observer frame as such. Similarly given the equation of geodesic or autoparallel line;

$$\frac{dx^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \quad (1)$$

it is clear that the torsion tensor $\Gamma_{[\beta\gamma]}^\alpha = \frac{1}{2}(\Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha)$ does not contribute since the differentials commute. This leads to interpretation of torsion as a non-propagating spin-contact interaction (see for example [15]). It will be shown below

however that a different interpretation is possible consistent with the findings of deAndrade and Pereira. A $ds^2 = 0$ component is ‘on the light cone’; this will be used as a vacant mathematical slot into which is plugged a spinorial description of electro-magnetism. This exploits the epicentre of the C-M theorem; the Lorentz group (and effectively also the Poincaré since all distances are zero for an ‘observer’ on a wave of light) is non-compact precisely because the speed of light is not in any observer’s frame. The analogue of a supersymmetry transformation then becomes the interconversion of two mathematical descriptions of one force; one spinorial and one bosonic which is called in the text a ‘translational’ procedure. This ‘translation’ is the weakest link in the theoretical construction but some concrete mathematical support for its consistency is supplied by studying the construction of stress-energy tensors from Lagrangians with anti-symmetric metrics in section VII. With this translation procedure the superpartner of the photon becomes - the photon!; but this superpartner is only detectable to an observer who is travelling at the speed of light so it never appears in experiments!

What are the problems? Different people might give different answers to this question but the following are a selection of the main obstacles to four-dimensional geometric unification of electro-magnetism and gravitation in the framework of general relativity;

1. For a dimensionless metric the scalar curvature R has dimension l^{-2} so that in four dimensions $\mathcal{L} = kR$ requires constant k to have dimension l^{-2} and so the theory is non-renormalisable.
2. How can the electro-magnetic vector potential A_μ be placed in the tensor $g_{\mu\nu}$ without spoiling its tensor character or destroying General Relativity; i.e. if we require the strong equivalence principle $g_{\mu\nu}; \phi = 0$ to hold. Note that this is related to the first problem because A_μ is dimensional with dimension l^{-1} .
3. How can the free-field stress-energy tensor be extracted from the metric? From the Lagrangian?
4. How can source terms be included in the metric? Again these must not spoil the qualities of the metric or destroy gravitation theory.
5. How can we couple the stress-energy tensors for gravitation and electro-magnetism into one equation relating to curvature?
6. Does the theory have scope for generalisation to the electro-weak interaction?
7. And what about quantisation?
8. The Coleman-Mandula theorem.

The paper is organised as follows; firstly there is a brief review of the history - particularly regarding homothetic curvature (à-là-Eddington) with which many readers may not be familiar. A metric is then defined and its consistency proven. It contains both a symmetric and an anti-symmetric part. A connection is then defined on the basis of the vanishing of the covariant derivative of the metric (vanishing non-metricity). The stress-energy tensor is then extracted by expanding the (homothetic) curvature tensor in the form of an Einstein equation. The metric is then redefined to accomodate source terms and the source-stress-energy tensor formed. Lastly the metric is applied to the Lagrangian formalism. Due to constraints of space the prospects for electro-weak-gravitation unification and explicit interaction terms are not discussed. The notation used throughout is perhaps somewhat traditional as the particular mathematical structure explored does not lend itself ideally to the notation of differential forms (e.g. the use of notation with ω connection one-forms, exterior derivative, exterior product etc; see for example Trautman [11]) because every index must be carefully tracked for anti-commutivity. The notation used is consistently applied and certainly familiar to anyone accustomed to the standard texts on General Relativity. Part of the work is an extension of ideas presented previously [31].

There is an extensive literature in this field but the ideas proposed in this paper are quite different from any previously published work to the best of my knowledge.

II. HISTORICAL BACKGROUND

It is instructive to consider the original efforts since the principles uncovered by the pioneers in the field underpin all efforts that followed. Theoretical efforts to form a unified description of gravity and electro-magnetism in the classical framework date from early this century beginning particularly with the work of Nördstrom, Weyl, Eddington, Einstein and Cartan. An excellent review is found in reference [5]. Weyl’s scheme [1] revolved around scale-transformations (gauge transformations). This work failed to provide a viable unified theory of gravitation and electro-magnetism, which was Weyl’s original intent, but subsequently proved very fruitful in other areas; Weyl is truly the father of the modern approach of gauge field theory.

In essence Weyl’s idea was to extend the geometric foundations of Riemannian geometry by allowing for scale transformations to vectors with parallel transport. This approach was criticised, particularly by Einstein, as being incompatible with observation; in particular it means that the physical properties of measuring rods and clocks depends upon their history. For example, two identical clocks, initially synchronised to run at the same rate in the

same inertial frame, will no longer do so if they are brought together at a later time into the same inertial frame having travelled through different paths in space-time according to Weyl's scheme.

Subsequently Eddington [26] attempted to extend Weyl's theory. In Eddington's theory, as in Weyl's, the connection forms the basic geometric object and is related to the electromagnetic potential A_β ;

$$\Gamma_{\alpha\beta}^\alpha = \Gamma_{\beta\alpha}^\alpha = \lambda.A_\beta \quad (2)$$

-for constant dimensionless lambda. Eddington then proceeds to form the anti-symmetric electro-magnetic tensor $F_{\beta\gamma}$ viz homothetic curvature;

$$\begin{aligned} R_{\alpha\beta\gamma}^\alpha &= \partial_\beta \Gamma_{\alpha\gamma}^\alpha - \partial_\gamma \Gamma_{\alpha\beta}^\alpha - \Gamma_{\phi\beta}^\alpha \Gamma_{\alpha\gamma}^\phi + \Gamma_{\phi\gamma}^\alpha \Gamma_{\alpha\beta}^\phi \\ &= F_{\beta\gamma} \end{aligned} \quad (3)$$

-where the two product terms in (3) have been equated to zero as is usually done in general relativity. In order to overcome the criticism of Weyl's theory with regard to measuring rods and clocks Eddington imposes an assumed metric condition on the curvature tensor (Eddington's 'natural gauge');

$$\phi g^{\alpha\beta} = R^{\alpha\beta}$$

where ϕ is a constant of dimension l^{-2} . This constraint effectively 'fine-tunes' the metric to the space-time curvature in an attempt to avoid the problem of measurement associated with the Weyl non-metric geometry.

There are a number of problems associated with the Eddington approach which emerged. In particular these involve the number of unknowns in the differential equations resulting from the use of the connection as the main geometric element (about 40) and higher-order derivative terms which arise in the theory. More generally, we can see an inconsistency with general relativity because the curvature tensor is equated viz Einstein's equation to the stress-energy tensor in G.R; the corresponding object in electro-magnetism is quadratic in the $F_{\mu\nu}$ not first-order in it. In fact it is the E-M stress-energy tensor which should appear on the R.H.S. of Einstein's equation contributing, at the very least, to the gravitational potential as it is a source of mass-energy equivalence. Also in the Weyl/Eddington theory the potential is identified with the connection; in G.R. it is identified with the metric. More recent studies using the Weyl/Eddington approach are found in refs [10].

Cartan appears to have been the first person to explore the possibility of theories involving torsion in the context of general covariance and classified possible theories on the basis of affine vs metric, (affine theories, such as Weyl's, are 'non-metric'), the presence or absence of rotation curvature (defined as present if $R_{\gamma\beta\alpha}^\alpha = R_{\beta\gamma\alpha}^\alpha \neq 0$), the presence or absence of homothetic curvature (present if $R_{\alpha\beta\gamma}^\alpha = -R_{\alpha\gamma\beta}^\alpha \neq 0$) and the presence or absence of torsion (present if $\Gamma_{\beta\gamma}^\alpha = -\Gamma_{\gamma\beta}^\alpha \neq 0$). Riemann-Cartan geometries have non-vanishing torsion. Many R-C geometries involve adding an anti-symmetric piece to the metric which is in some way related to the Maxwell tensor $F_{\alpha\beta}$;

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + \lambda F_{\alpha\beta}$$

where $h_{\alpha\beta}$ is the gravitational potential in the weak field approximation. It is now understood that torsion is related to translations [9] (torsion 'breaks' parallelograms - it is also related to the theory of crystal dislocations [12]) whilst rotation curvature is related to rotations. This situation is somewhat paradoxical (as has been noted) since the (non-compact) Poincaré group, the 'gauge' group for gravitation, is the group of translations. In the treatment given here we will see the reverse by embedding electro-magnetism (a U1 or compact rotation group symmetry) in torsion; which has the geometry of translations not rotations! An attempt to give a geometric interpretation to these apparent contradictions will be given in the discussion section when all the geometric machinery needed has been developed.

At about the same time as Eddington published his theory Kaluza published his five-dimensional version of gravitational-electromagnetic unification [14].

The fifth dimension in the Kaluza-Klein theory is a periodic space which spontaneously compactifies. (More contemporary versions of the Kaluza-Klein geometry attempt to extend the compactified space to higher dimensions to accomodate the $S.U._3 \times S.U._2 \times U_1$ standard model; see ref. [14] for examples. See also [16]). The Kaluza-Klein theory is appealing for a number of basic reasons. Often overlooked but of basic importance is the fact that the metric in the theory parallels general relativity by containing the potential of the theory; most alternative attempts at unification have attempted to site the vector potential A_μ in the connection. However, the Kaluza-Klein theory remains a five-dimensional theory and ideally we would like a four-dimensional theory; the universe is not observed to be anything other than four dimensional so we have no empirical evidence for the extra dimensions. In addition to the Kaluza-Klein theory there are numerous theories identifying an anti-symmetric component of the metric with the Maxwell tensor $F^{\mu\nu}$ such as Einstein's unified field theory [7] and later contributions to U_4 theory (with torsion) development from

Sciama [17] and Kibble [18] and others. More contemporary approaches include 3D Riemann-Cartan geometry with Yang-Mills fields ([19] [22] and contained references). Lunev [20] develops an Einstein equation for Yang-Mills fields but the approach used differs from the one employed in this paper by placing the potential in the connection. Garcia deAndrade and Hammond [23] employ the Eddington approach of equating homothetic curvature and the Maxwell tensor and interpret massive torsion quanta as massive photons. More recently Unzicker [21] has explored teleparallel geometry and electromagnetism although the latter approach is very different from the one presented here.

Attempts to embed a description of electro-magnetism in general covariant theory have difficulty because, unlike gravity which can be ‘transformed away’ *locally* in a free-fall frame, the electro-magnetic field cannot be ‘transformed away’ by a Lorentz transformation. There appears however to be a loophole that can be exploited here without explicitly breaking Lorentz invariance; that of working on the light-cone itself - in a non-observer frame. What this means will be discussed below.

It has been pointed out [8] that gauge freedom in Proca’s equation for $m = 0$ precludes torsion in electro-magnetism (but not in the case $m \neq 0$ for a spin-1 field). Thus theories with torsion in electro-magnetism frequently imply photon mass [15]. However, in the derivation presented it will shown that a constraint emerges from the connection which permits the description of torsion in electro-magnetism with massless photons; possibly providing an explanation for the apparent conflict between the results of Hehl et. al. and deAndrade et. al. alluded to above.

III. METRIC

Consider the following metric;

$$\begin{pmatrix} +I_4 & \frac{i}{2}\sigma_{01} & \frac{i}{2}\sigma_{02} & \frac{i}{2}\sigma_{03} \\ \frac{i}{2}\sigma_{10} & -I_4 & \frac{i}{2}\sigma_{12} & \frac{i}{2}\sigma_{13} \\ \frac{i}{2}\sigma_{20} & \frac{i}{2}\sigma_{21} & -I_4 & \frac{i}{2}\sigma_{23} \\ \frac{i}{2}\sigma_{30} & \frac{i}{2}\sigma_{31} & \frac{i}{2}\sigma_{32} & -I_4 \end{pmatrix} = I_4 \cdot \eta_{\alpha\beta} + \frac{i}{2}\sigma_{\alpha\beta} \quad (4)$$

where $\sigma^{\alpha\beta} = \frac{i}{2} [\gamma^\alpha, \gamma^\beta]$ and introducing the notation $\sigma'^{\alpha\beta} = \frac{1}{2}\sigma^{\alpha\beta}$ we have (noting in the sum $\sigma'_{\alpha\phi}\sigma'^{\phi}_{\beta}$, ϕ can only take two values as α and β are different valued);

$$\begin{aligned} g_{\alpha\phi}\bar{g}^\phi_{\beta} &= \left(I_4 \cdot \eta_{\alpha\phi} + i\sigma'_{\alpha\phi}\right) \left(I_4 \cdot \eta^\phi_{\beta} - i\sigma'^{\phi}_{\beta}\right) \\ &= \left(I_4 \cdot \eta_{\alpha\beta} + i\sigma'_{\alpha\beta}\right) = g_{\alpha\beta} \end{aligned} \quad (5)$$

where \bar{g} is the complex-conjugate transpose (\dagger) or ‘dual’ metric viz;

$$\left(\sigma'_{\alpha\beta}\right)^\dagger = \frac{-i}{4} [\gamma_\alpha, \gamma_\beta]^\dagger = \sigma'^{\alpha\beta} \quad (6)$$

so that

$$(g_{\alpha\beta})^\dagger = \bar{g}^{\alpha\beta} = \left(I_4 \cdot \eta^{\alpha\beta} - i\sigma'^{\alpha\beta}\right) = g^{\beta\alpha} \quad (7)$$

Note that;

$$g_{\alpha\phi} g_\beta^\phi = g_{\alpha\beta} \quad (8)$$

is exceedingly constraining. The general solution for the Lorentz tangent space metric (+,-,-,-) includes an anti-symmetric component. Now the Dirac gamma matrices do not transform as four-vectors. However, the sigma matrices formed from them transform as tensors;

$$\frac{i}{2}\sigma_{\alpha\beta} = \frac{-1}{4}[\gamma_\alpha, \gamma_\beta]_- \quad (9)$$

which transforms as a true tensor. It is the antisymmetric version of the fundamental tensor which can be defined as;

$$\eta_{\alpha\beta} = \frac{1}{2}\{\gamma_\alpha, \gamma_\beta\}_+ \quad (10)$$

Which dictates the solution space for (8) for which metric (4) and its dual are the general solution. Of course the anti-symmetric piece does not span the space S.O.(3,1) at all but is confined to the end point of boosts - i.e. $ds^2 = 0$ - *which is not in the group*. The S.O.(3,1) group is non-compact precisely because the velocity of light, the end-point of boosts, is not in the group. What metric (4) does is insert a new part of the metric onto the light cone. The transformations associated with using the sigma matrix as the fundamental tensor constitute a new group confined to be on the light cone itself. Let us call this group C.O.(3,1). We will study its properties later but we note that;

$$\text{S.O.}(3,1) \cup \text{C.O.}(3,1)$$

will be (trivially) topologically compact.

Of all possible anti-symmetric pieces which may be added to the metric $\eta_{\alpha\beta}$ only one possible term satisfies both the consistency equation (eq.(8)) and Lorentz covariance and it is, not suprisingly, the fundamental tensor formed by substituting the commutator for the anticommutator of the gamma matrices. There is a profound underlying symmetry in this which I do not fully understand. It should be apparent, however, that the metric (4) is highly non-trivial and unique.

From now on I will drop the prime on the sigma matrices it being understood that *all* subsequent expressions containing sigma matrices are of the primed form (i.e. the factor of $\frac{1}{2}$ is absorbed into the definition). The off-diagonal elements of this metric are 4x4 matrices so the metric is 16x16. Diagonal identities are 4x4 and each space-time index can be multiplied into a 4x4 identity to couple to the metric. Thus although the metric is now a 16x16 matrix we still only need four parameters to describe the space which is, physically and mathematically, still therefore four dimensional; as implied by the R.H.S. of (4) where the Greek indices range over 4 values. Normally we expect the antisymmetric σ matrices to contract against spinors - but here they will be contracted against commuting co-ordinates. Apart from a brief comment at the end of the paper I will not deal further with the issue of co-ordinates. The theory is constructed in a co-ordinate independent fashion. Contraction with the dual metric produces the scalar identity (omitting the implicit matrix multiplier of I_4);

$$g_{\alpha\beta} \bar{g}^{\beta\alpha} = +4 - 3 = +1$$

although the metric is not invertible as a Kronecker delta. Consequently care is required in raising and lowering indices (see below). The apparent lack of invertibility of the metric causes no problem; the different parts of the metric label different fields and indices for each field are appropriately raised and lowered with each field's respective metric with cross terms generating interactions. Thus when the physical content of the theory is inserted the metric is well behaved. Note that Lorentz scalars such as $P^2 = m^2$ and ds^2 are still invariant under this metric. This is important because we wish to construct a theory which preserves physical measurements (one of the main criticisms of the Weyl approach was that it did not preserve physical measurement invariants in different regions of space [5]). To obtain a dynamical theory we will require the derivatives of the off-diagonal anti-symmetric part of metric (4) to be non-vanishing. To facilitate this we introduce a parameter $|P|$, with non-vanishing space-time derivatives and modulus unity, and incorporate $|P|$ into the sigma matrices;

$$\sigma_{\alpha\beta} \equiv |P| \sigma_{\alpha\beta} \quad (11)$$

We will take $|P|$ as a one-parameter group $|P| = e^{\pm i k \cdot x}$ where k^μ is the photon four-momentum and x_μ the space-time four-vector in units $\hbar = c = 1$. We will see below that consistency of the metric can be maintained with this added phase-factor.

IV. CONNECTION COEFFICIENTS

We will require the vanishing of the covariant derivative of the metric;

$$g_{\alpha\beta;\gamma} = (I_4 \cdot \eta_{\alpha\beta} + i|P| \sigma_{\alpha\beta})_{;\gamma} = 0. \quad (12)$$

We consider only a free-fall frame in which the derivatives of the diagonal elements of the metric vanish; the derivatives of off-diagonal elements however will be non-vanishing in this frame (as we shall see this applies when there is an electro-magnetic field present). Now

$$(|P| \sigma_{\alpha\beta})_{;\gamma} |P| \sigma^\phi_{\beta} \approx i |P|_{;\gamma} \sigma_{\alpha\beta} = i (|P| \sigma_{\alpha\beta})_{;\gamma} \quad (13)$$

(\approx here means equal up to a (local) phase factor). From now on the parameter $|P|$ will be absorbed into the definition of the sigma matrices ($|P| \sigma_{\alpha\beta} \equiv \sigma_{\alpha\beta}$) in all expressions.

For both indices downstairs I will use $e^{+ik \cdot x}$ and for the dual with both indices upstairs the $e^{-ik \cdot x}$ so that

$$(g_{\alpha\beta})^\dagger = \bar{g}^{\alpha\beta}$$

Raising or lowering a single index thus eliminates the phase factor as a consequence.

Writing and defining the covariant derivative of the asymmetric part of the metric with three different labellings;

$$g_{\alpha\beta;\gamma}^A = g_{\alpha\beta,\gamma}^A - \Gamma_{\gamma\alpha}^\phi i\sigma_{\phi\beta} - \Gamma_{\gamma\beta}^\phi i\sigma_{\alpha\phi} \quad (14)$$

$$g_{\gamma\beta;\alpha}^A = g_{\gamma\beta,\alpha}^A - \Gamma_{\alpha\gamma}^\phi i\sigma_{\phi\beta} - \Gamma_{\alpha\beta}^\phi i\sigma_{\gamma\phi} \quad (15)$$

$$g_{\alpha\gamma;\beta}^A = g_{\alpha\gamma,\beta}^A - \Gamma_{\beta\alpha}^\phi i\sigma_{\phi\gamma} - \Gamma_{\beta\gamma}^\phi i\sigma_{\alpha\phi} \quad (16)$$

$$\Gamma_{\alpha\beta}^\phi = \frac{1}{2} (g_{\alpha\epsilon,\beta} + g_{\epsilon\beta,\alpha} - g_{\alpha\beta,\epsilon}) \bar{g}^{\epsilon\phi} \quad (17)$$

using (17) and raising indices with the anti-symmetric part of (7) (we have a choice in this situation of raising indices either with the symmetric part of the metric *or*, the antisymmetric part or both. For the free-field (no interactions) we require only the anti-symmetric part of the metric which means, because the derivatives of the diagonal part vanish, we are effectively working with a purely anti-symmetric metric in the derivation), we have (16) + (15) - (14) gives;

$$\begin{aligned} g_{\alpha\gamma,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma} \\ = i\sigma_{\alpha\gamma,\beta} + i\sigma_{\gamma\beta,\alpha} - i\sigma_{\alpha\beta,\gamma} \end{aligned} \quad (18)$$

provided we define contractions on the derivative index *from its right* as;

$$\sigma_{\alpha\beta},^\phi \sigma_{\phi\gamma} = +i\sigma_{\alpha\beta,\gamma} \quad (19)$$

showing (17) is consistent. Notice that the connection so defined is antisymmetric in its lower two indices; this is a torsion connection. Also note that although $\Gamma_{\alpha\beta}^\phi = -\Gamma_{\beta\alpha}^\phi$ we cannot use this to interchange indices and sum connection components; $\sigma_{\alpha\beta,\epsilon} \neq -\sigma_{\epsilon\beta,\alpha}$ for individual components.

V. HOMOTHEIC CURVATURE; $R_{\alpha\beta\gamma}^\alpha$

In contrast to gravitational theory the homothetic curvature is non-zero. We contract over the first upper and first lower index of the curvature tensor [25] [26]

$$R_{\alpha\beta\gamma}^\alpha = \partial_\beta \Gamma_{\alpha\gamma}^\alpha - \partial_\gamma \Gamma_{\alpha\beta}^\alpha - \Gamma_{\phi\beta}^\alpha \Gamma_{\alpha\gamma}^\phi + \Gamma_{\phi\gamma}^\alpha \Gamma_{\alpha\beta}^\phi \quad (20)$$

which is anti-symmetric in its two uncontracted indices. For a theory of electro-magnetism we require first derivatives of the potential terms. Thus we are interested in the product of connection coefficients in (20) which, for the case at hand, are non-vanishing in the presence of metric (12). We will later see that the other two terms with second derivatives of the metric cancel in (20). Now consider the Bianci identity;

$$R_{\alpha\beta\gamma;\delta}^\alpha + R_{\alpha\delta\beta;\gamma}^\alpha + R_{\alpha\gamma\delta;\beta}^\alpha = 0 \quad (21)$$

and contract with the full metric;

$$(R_{\alpha\beta\gamma;\delta}^\alpha + R_{\alpha\delta\beta;\gamma}^\alpha + R_{\alpha\gamma\delta;\beta}^\alpha) \bar{g}^{\beta\gamma} = 0 \quad (22)$$

relabelling and using the fact that a product of symmetric and anti-symmetric parts with the same indices is zero we obtain;

$$\begin{aligned} -i (R_{\alpha\beta\gamma;\delta}^\alpha - 2R_{\alpha\beta\delta;\gamma}^\alpha) \sigma^{\beta\gamma} &= 0 \\ \frac{1}{2} R_{;\delta}^A - R_{\delta;\gamma}^\gamma &= 0 \end{aligned} \quad (23)$$

where the scalar $-i R_{\alpha\beta\gamma;\delta}^\alpha \sigma^{\beta\gamma} \equiv R_{;\delta}^A$ and indices are contracted in the tensor part. Finally relabelling and raising indices with the *symmetric* part of the metric we obtain;

$$\left(\frac{1}{2} \eta^{\phi\delta} R_A - R^{\delta\phi} \right)_{;\delta} = 0 \quad (24)$$

Although this equation appears identical to Einstein's equation it contains very different information. Note also that in (24) I have employed the opposite sign convention than is usual in Einstein's equation. This is a reasonable assertion since the gravitational potential is unbounded from below whilst the electro-magnetic potential for a charged object is unbounded from above as $r \rightarrow 0$ so we expect curvatures which enter with opposite sign.

VI. CALCULATION OF ELECTRO-MAGNETIC TORSION

Using (7), (17) and (19) we have;

$$\Gamma_{\alpha\beta}^\alpha = \frac{1}{2} g_{\alpha\epsilon,\beta} \bar{g}^{\epsilon\alpha} = -\Gamma_{\beta\alpha}^\alpha \quad (25)$$

and thus;

$$\begin{aligned} & \partial_\beta \Gamma_{\alpha\gamma}^\alpha - \partial_\gamma \Gamma_{\alpha\beta}^\alpha \\ &= \frac{1}{2} g_{\alpha\epsilon,\gamma,\beta} \bar{g}^{\epsilon\alpha} + \frac{1}{2} g_{\alpha\epsilon,\gamma} \bar{g}^{\epsilon\alpha}_{,\beta} - \frac{1}{2} g_{\alpha\epsilon,\beta,\gamma} \bar{g}^{\epsilon\alpha} - \frac{1}{2} g_{\alpha\epsilon,\beta} \bar{g}^{\epsilon\alpha}_{,\gamma} \\ &= \frac{1}{2} g_{\alpha\epsilon,\gamma} \bar{g}^{\epsilon\alpha}_{,\beta} - \frac{1}{2} \bar{g}_{\epsilon\alpha,\beta} g^{\alpha\epsilon}_{,\gamma} \\ &= 0 \end{aligned} \quad (26)$$

where the last line follows because the $g_{\epsilon\alpha}$'s commute as do the derivative indices. Hence the components containing derivatives of the connection of (20) vanish and we have;

$$\begin{aligned} -i R_{\alpha\gamma\beta}^\alpha \sigma^{\gamma\beta} &= -i \left(\Gamma_{\phi\beta}^\alpha \Gamma_{\alpha\gamma}^\phi - \Gamma_{\phi\gamma}^\alpha \Gamma_{\alpha\beta}^\phi \right) \sigma^{\gamma\beta} \\ &= -2i \Gamma_{\phi\beta}^\alpha \Gamma_{\alpha\gamma}^\phi \sigma^{\gamma\beta} \\ &= -\frac{i}{2} \begin{pmatrix} i\sigma_{\phi,\beta}^\alpha & +i\sigma_{\beta,\phi}^\alpha & -i\sigma_{\phi,\beta}^{\alpha,\phi} \\ \text{(A)} & \text{(B)} & \text{(C)} \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} +i\sigma_{\alpha,\gamma}^\phi & +i\sigma_{\gamma,\alpha}^\phi & -i\sigma_{\alpha\gamma}^{\phi,\phi} \\ \text{(D)} & \text{(E)} & \text{(F)} \end{pmatrix} \sigma^{\gamma\beta} \end{aligned} \quad (27)$$

Now consider the product involving terms (B) and (E);

$$\begin{aligned} -\frac{i}{2} i\sigma_{\beta,\phi}^\alpha i\sigma_{\gamma,\alpha}^\phi \sigma^{\gamma\beta} &= -\frac{1}{2} \sigma^{\alpha\beta,\phi} \sigma_{\phi\beta,\alpha} \\ &\stackrel{(\text{def.})}{\equiv} -\frac{1}{2} \partial^\phi A^\alpha \partial_\alpha A_\phi \end{aligned} \quad (28)$$

The last line involves a contraction over β and a dimensional transmutation to define the A field. This definition is the 'translation' alluded to earlier in the paper and is discussed extensively later in the paper. Similarly the product of terms (C) and (F) of eq (27) gives an identical $-\frac{1}{2} \partial^\phi A^\alpha \partial_\alpha A_\phi$. For (B).(F) and (C).(E) of eq. (27) we obtain;

$$-\frac{i}{2} \sigma_{\beta,\phi}^\alpha \sigma_{\alpha\gamma}^{\phi,\phi} \sigma^{\gamma\beta} \equiv +\frac{1}{2} \partial^\phi A^\alpha \partial_\phi A_\alpha$$

The products (B).(D), (C).(D) are zero because the $\sigma^{\gamma\beta}$ commutes past the derivative index of $\sigma_{\alpha,\gamma}^\phi$ and hence contracts with opposite sign on the γ and β . The products (A)(E) and (A)(F) are also zero for the same reason (to see this first anti-commute the two matrices; $\sigma_{\phi,\beta}^\alpha \sigma_{\gamma,\alpha}^\phi$ - note also that two sigma's with dummy contracted indices anti-commute if, with relabelling, there is only one index interchange on the sigma's - otherwise they commute). The last product term ((A).(D) in eqn. (27)) is zero because the derivative indices commute. Hence we have for the scalar part of (24) summing contributions R equals;

$$- \partial^\phi A^\alpha \partial_\alpha A_\phi + \partial^\phi A^\alpha \partial_\phi A_\alpha = \frac{1}{2} F^{\phi\alpha} F_{\phi\alpha} \quad (29)$$

The tensor part of (24) is similarly calculated. A subtlety however arises with regard to translations into forms like (28) because of the anti-symmetry of the tensor piece R_γ^δ . I will calculate the terms first, impose anti-symmetry on the translation into the A-field terms ‘by hand’ and then explain the meaning of the translation later in the text;

$$\begin{aligned} & + 2i R_{\alpha\delta\beta}^\alpha \sigma^{\gamma\beta} = +2i \left(\Gamma_{\phi\beta}^\alpha \Gamma_{\alpha\delta}^\phi - \Gamma_{\phi\delta}^\alpha \Gamma_{\alpha\beta}^\phi \right) \sigma^{\gamma\beta} \\ & = +\frac{i}{2} \left(\begin{matrix} \text{(A)} & \text{(B)} & \text{(C)} \\ i\sigma_{\phi,\beta}^\alpha & +i\sigma_{\beta,\phi}^\alpha & -i\sigma_{\phi\beta}^\alpha \end{matrix} \right) \cdot \\ & \quad \left(\begin{matrix} \text{(D')} & \text{(E')} & \text{(F')} \\ i\sigma_{\alpha,\delta}^\phi & +i\sigma_{\delta,\alpha}^\phi & -i\sigma_{\alpha\delta}^\phi \end{matrix} \right) \sigma^{\gamma\beta} \\ & \quad -\frac{i}{2} \left(\begin{matrix} & \text{(A')} & \text{(B')} & \text{(C')} \\ i\sigma_{\phi,\delta}^\alpha & +i\sigma_{\delta,\phi}^\alpha & -i\sigma_{\phi\delta}^\alpha \end{matrix} \right) \cdot \\ & \quad \left(\begin{matrix} i\sigma_{\alpha,\beta}^\phi & +i\sigma_{\beta,\alpha}^\phi & -i\sigma_{\alpha\beta}^\phi \\ \text{(G)} & \text{(H)} & \text{(I)} \end{matrix} \right) \sigma^{\gamma\beta} \end{aligned} \quad (30)$$

The only products which are zero in (30) are (A).(D') and (A').(G). Relabelling dummies shows that the remaining products in (30) anti-commute. For (B').(H) we have;

$$+\frac{i}{2} \sigma_{\delta,\phi}^\alpha \sigma_{\beta,\alpha}^\phi \sigma^{\gamma\beta} = -\frac{1}{2} \sigma^{\phi\gamma}_{,\alpha} \sigma^\alpha_{\delta,\phi} \quad (31)$$

which sums with (B).(E'). Analogous contributions arise from (C).(F') and (C').(I). The crossed term (B).(F'), (B').(I), (C).(E') and (C').(H), each give;

$$\frac{i}{2} \sigma_{\beta,\phi}^\alpha \sigma_{\alpha\delta}^\phi \sigma^{\gamma\beta} \equiv +\frac{1}{2} \partial^\phi A^\gamma \partial_\phi A_\delta \quad (32)$$

For similar reasons the product (A).(E') gives

$$-1/2 \partial^\gamma A^\alpha \partial_\alpha A_\delta$$

and similarly for (A).(F'), (B').(G) and (C').(G). Products (B).(D'), (C).(D'), (A').(H) and (A').(I) are easily evaluated and each gives $-\frac{1}{2} \partial^\phi A^\gamma \partial_\delta A_\phi$. To translate (31) the α contraction on the indices delivers a $+i\partial_\delta$ and the ϕ contraction a $-i\partial^\gamma$; the -i sign because with relabelling it can be seen that the two matrices

$$\sigma^{\phi\gamma}_{,\alpha} \sigma^\alpha_{\delta,\phi}$$

anti-commute. Hence we obtain;

$$-\frac{1}{2} \sigma^{\phi\gamma}_{,\alpha} \sigma^\alpha_{\delta,\phi} \equiv +\frac{1}{2} \partial^\gamma A^\phi \partial_\delta A_\phi \quad (33)$$

Summing the non-zero components of the tensor part we have;

$$\begin{aligned} + 2i R_{\alpha\delta\beta}^\alpha \sigma^{\gamma\beta} &= -2R_\delta^\gamma = +2R_\delta^\gamma \\ &\equiv -2\partial^\gamma A^\alpha \partial_\alpha A_\delta + 2\partial^\alpha A^\gamma \partial_\alpha A_\delta \\ &\quad + 2\partial^\gamma A^\alpha \partial_\delta A_\alpha - 2\partial^\alpha A^\gamma \partial_\delta A_\alpha \\ &\equiv 2F^{\gamma\alpha} F_{\delta\alpha} \end{aligned} \quad (34)$$

Raising indices with the symmetric part of the metric we finally obtain the traceless electro-magnetic stress-energy tensor;

$$-\frac{1}{\kappa^2} R^{\delta\gamma} + \frac{1}{2\kappa^2} \eta^{\gamma\delta} R = F^{\gamma\alpha} F_\alpha^\delta + \frac{1}{4} \eta^{\gamma\delta} F^{\mu\nu} F_{\mu\nu} \quad (35)$$

In forming equation (35) I have replaced the equivalence relation (\equiv) by an = sign and a dimensional constant κ^{-2} (the gravitational coupling constant). This is discussed in section IX.

The derivation of the traceless gauge-invariant free-field stress-energy tensor equated to the Einstein-like equation is something of a mathematical miracle. There must be exactly the right number and type of non-zero pieces to construct the tensor and the factor of 2 difference between the scalar R and the tensor $R^{\delta\gamma}$ on the L.H.S. of eq.(35) gets translated into an effective factor of 4 difference on the R.H.S. only because of the spinorial representation used and the translation procedure. This is actually a non-trivial result. I suspect it is the only way a traceless gauge-invariant F.F. tensor can be extracted from a standard Lagrangian.

The issue of the anti-symmetry in δ and γ of (34) is discussed below.

VII. DISCUSSION OF HOMOTHETIC CURVATURE

The above derivation effectively eliminates the symmetric part of the metric. Applying the anti-symmetry constraint $\mu \neq \nu$ for a purely anti-symmetric metric;

$$\bar{g}_A^{\mu\nu} = -i\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]_- = \frac{1}{2}\gamma^\mu\gamma^\nu_{(\mu \neq \nu)} \quad (36)$$

so, using (13);

$$\begin{aligned} \sigma^{\nu\mu, \delta} \sigma_{\nu\delta, \mu} &= \frac{1}{4} \partial^\delta (\gamma^\mu \gamma^\nu) \partial_\mu (\gamma_\nu \gamma_\delta) \\ &\approx \frac{1}{4} (\partial^\delta \gamma^\mu) \gamma^\nu \gamma_\nu (\partial_\mu \gamma_\delta) = (\partial^\delta \gamma^\mu) (\partial_\mu \gamma_\delta) \end{aligned} \quad (37)$$

which identifies the A^μ field as a γ^μ 4x16 matrix $|P|\gamma^\mu \equiv A^\mu$ transforming as a *vector* under the 16x16 anti-symmetric metric (36); *with respect to commuting co-ordinates* (note; use of commuting co-ordinates implicit in the derivation of results - note also that (13) implies that the A field is only defined up to a local phase). Normally an infinitesimal rotation is given by

$$\delta x^i = \epsilon^{ij} \eta_{jk} x^k = \epsilon^i{}_k x^k$$

where ϵ^{ij} is antisymmetric. Now R^γ_δ in (34) is anti-symmetric but under g^A an infinitesimal rotation is given by; $\delta x^i = s^{ik} g^A_{kj} x^j$ where s^i_j is symmetric thus variation of the Lagrangian [28] (for generic field ϕ) will give;

$$0 = s_{\mu\nu} \partial_\rho \left[\frac{\delta \mathcal{L}}{\delta \partial_\rho} (\partial^\mu \phi x^\nu + \partial^\nu \phi x^\mu) - g^{\rho\nu} x^\mu \mathcal{L} - g^{\rho\mu} x^\nu \mathcal{L} \right]$$

with the divergence of the conserved current;

$$\partial_\rho \mathcal{M}^{\rho, \mu\nu} = \mathcal{T}^{\mu\nu} + \mathcal{T}^{\nu\mu}$$

which is zero if the stress-energy tensor $T^{\mu\nu}$ is *anti-symmetric* under g^A (in other words, in the framework of an anti-symmetric metric the stress-energy tensor must be *anti-symmetric* to obtain conservation of angular momentum - this is the opposite to the situation with a purely symmetric metric where the stress-energy tensor must be *symmetric* to conserve angular momentum).

Effectively we have a choice of description; (1) we can describe the A field as a conventional vector with symmetric metric in commuting co-ordinates, or (2) as a ' γ ' vector with anti-symmetric metric in commuting co-ordinates. We know from (36) that A^μ must transform as a 4 vector under the space-time metric. Raising indices in (35) with the diagonal part of (12) implies we revert to description (1) instead of (2) where the A^μ is no-longer a γ^μ vector but a simple 4-vector transforming under symmetric metric and $T^{\gamma\delta}$ is instead symmetric because angular momentum conservation must be present regardless of the choice of description. However, this of course means that we must equivalently substitute a *symmetric* $R^{\gamma\delta}$ in eq (35) for the anti-symmetric value that arises in eq (34). This relates back to the A-field definitions like eq (33) the notation of which is appropriate for a symmetric term. It is the L.H.S. of eq (33), and analogous contributions, which should properly be summed to form the antisymmetric object R^γ_δ in eq (34); the conventional A-field definition (in commuting co-ordinates) is only appropriate when we translate to the symmetric objects (i.e. eq(35)). I have introduced the A-field notation (eq(28), eq (33) etc) early as this facilitates comprehension and also demonstrates that there is a consistent mathematical method for performing the translation. It must be noted however that there is always an inherent choice of sign on the tensor part when we perform a translation between an anti-symmetric and a symmetric object; this is the price we pay for working in a

spinorial representation against an anti-symmetric metric which becomes a representation *up to a sign*. For example eq (24) can be rewritten as; $(\frac{1}{2}\eta^{\phi\delta}R_A + R^{\phi\delta})_{;\delta} = 0$ where the tensor part now has opposite sign. It can however be argued that the same traceless stress-energy tensor will result since we can choose a sign from the residual phase factor from the index raising operation in eq(30) (the phase can be made to vanish for the scalar R_A).

Lastly in this section note that the Lagrangian for the free-field is now given by the scalar curvature;

$$\mathcal{L} = \frac{1}{\kappa^2} \sqrt{|g|} \lambda R = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \quad (38)$$

where λ is a normalisation constant. Because the representation effectively normalises the A field the norm of g is a constant and can be absorbed into the κ . Just how the κ^2 gravitational constant is absorbed into the free-field is discussed later in the text.

VIII. ROTATION CURVATURE AND SOURCE TERMS

Re writing eq.(20) for conventional curvature we have;

$$R_{\gamma\beta\alpha}^{\alpha} = \partial_{\beta} \Gamma_{\gamma\alpha}^{\alpha} - \partial_{\alpha} \Gamma_{\gamma\beta}^{\alpha} - \Gamma_{\phi\beta}^{\alpha} \Gamma_{\gamma\alpha}^{\phi} + \Gamma_{\phi\alpha}^{\alpha} \Gamma_{\gamma\beta}^{\phi} \quad (39)$$

In General Relativity the symmetric metric feeds into the rotation curvature viz the symmetric connection and defines the stress-energy tensor viz Einstein's equation. It is relatively easy to prove that the rotation curvature is zero for the component of metric (12) that is on the light cone (i.e. the antisymmetric part of the metric). We identify the rotation curvature with material sources; i.e. particles with mass and these sources should be identified with the symmetric part of the metric. The phase factor identified with the $ds^2 = 0$ part of the metric (the A_{μ} field) contains an implicit factor of $\hbar = 1$ and thus imply a 'waviness' to space-time structure at the quantum scale. It is this wave-like structure of small-scale space-time that has replaced the fifth dimension of Kaluza-Kline theory; the space-time structure itself has been given the properties of the harmonic oscillator.

Thus in order to obtain particle sources for the theory we must now modify the small-scale structure of space-time for the symmetric part of the metric. The appropriate phase factor will now be based on particle momentum and the metric takes the form ($\hbar = c = 1$);

$$g_{\mu\nu} = e^{ip \cdot x} I_4 \cdot \eta_{\mu\nu} + i e^{ik \cdot x} \sigma_{\mu\nu} \quad (40)$$

where p^{α} is the source four-momentum and k^{α} is the photon four-momentum.

The first two components of the expansion of the rotation curvature (R.H.S. 39) are zero since;

$$\partial_{\gamma} (\partial_{\beta} e^{ip \cdot x}) e^{-ip \cdot x} = ip_{\beta} \partial_{\gamma} (e^{ip \cdot x} e^{-ip \cdot x}) = 0$$

so there is no interference with gravitation at the level of the E.M. sources and we have;

$$R_{\gamma\beta\alpha}^{\alpha} = -\Gamma_{\phi\beta}^{\alpha} \Gamma_{\gamma\alpha}^{\phi} + \Gamma_{\phi\alpha}^{\alpha} \Gamma_{\gamma\beta}^{\phi} \quad (41)$$

For the source the metric is symmetric and the connection takes the usual symmetric form. We may take it as identical to (17) with the anti-symmetric part omitted and thus we obtain (for notational convenience dropping the I_4 and absorbing the phase-factor into the definition of η in an analogous manner as was done with the σ matrices);

$$\begin{aligned} 4R_{\gamma\beta\alpha}^{\alpha} &= -(\eta_{\phi, \beta}^{\alpha} + \eta_{\beta, \phi}^{\alpha} - \eta_{\phi\beta, \alpha}^{\alpha}) e^{-ip \cdot x} \\ &\quad (\eta_{\gamma, \alpha}^{\phi} + \eta_{\alpha, \gamma}^{\phi} - \eta_{\gamma\alpha, \phi}^{\phi}) e^{-ip \cdot x} \\ &\quad + (\eta_{\phi, \alpha}^{\alpha} + \eta_{\alpha, \phi}^{\alpha} - \eta_{\phi\alpha, \alpha}^{\alpha}) e^{-ip \cdot x} \\ &\quad (\eta_{\gamma, \beta}^{\phi} + \eta_{\beta, \gamma}^{\phi} - \eta_{\gamma\beta, \phi}^{\phi}) e^{-ip \cdot x} \\ &= +2 (\partial_{\gamma} e^{ip \cdot x}) e^{-ip \cdot x} (\partial_{\beta} e^{ip \cdot x}) e^{-ip \cdot x} \\ &\quad - 2\eta_{\gamma\beta} (\partial_{\phi} e^{ip \cdot x}) e^{-ip \cdot x} (\partial^{\phi} e^{ip \cdot x}) e^{-ip \cdot x} \end{aligned} \quad (42)$$

from which we obtain;

$$R_{\gamma\beta\alpha}^{\alpha} = -\frac{1}{2} p_{\gamma} p_{\beta} + \frac{m_o^2}{2} \eta_{\gamma\beta} \quad (43)$$

where m_o^2 is the square of the rest mass and

$$R = R_{\gamma\beta\alpha}^\alpha \eta^{\beta\gamma} = +\frac{3}{2}m_o^2 \quad (44)$$

so

$$-R_{\gamma\beta\alpha}^\alpha + \frac{1}{2}\eta_{\gamma\beta}R = \frac{m_o^2}{2} \left(U_\gamma U_\beta + \frac{1}{2}\eta_{\gamma\beta} \right) \quad (45)$$

where $U_\gamma = \frac{dx_\gamma^p}{d\tau}$ where τ is the proper time and p denotes the particle position. I have suppressed the phase factors associated with the symmetric metric contraction in eq(44) because what is being performed here is a translation to a classical description of a point particle. We will eventually add a factor of dimension l^3 to the symmetric part of metric (40) in order to create a Lagrangian density of the appropriate dimension. With translation a factor of l^{-3} will appear in the fields. In anticipation of this we add a factor of dimension l^{-3} to obtain a translation to a classical particle description with position $x(\tau)$ and use;

$$\begin{aligned} d^3(x) &= \int d\tau \delta(x^o - x_p^o(\tau)) \delta^3(x^i - x_p^i(\tau)) \\ &= \frac{d\tau}{dx^o} \delta^3(x^i - x_p^i(\tau)) \end{aligned} \quad (46)$$

so that eq(45) finally gives (dropping the factor of 1/2 which is analogous to the zero-point energy of an harmonic oscillator);

$$\frac{m_o}{2} \lambda T_{\gamma\beta} = \frac{m_o^2}{2} U_\gamma U_\beta \frac{d\tau}{dx^o} \delta^3(x^i - x_p^i(\tau)) \quad (47)$$

which is the correct form for the classical stress-energy tensor for a point-particle with unit charge [27]. The three-dimensional delta function has been substituted for the fields in the classical description (c.f eq(57)). (The delta function, in the classical limit that the space-time spread for the particle approaches a point, behaves as the inverse of the irreducible metric - see eqs (48),(51) and (52) - thus the use of the delta function is only valid in the ‘classical limit’ and not in a quantum description in which case the irreducible metric can not be treated as the inverse of a delta function). Lambda is a constant to be determined. Note that I have used the same sign convention for the particle stress-energy tensor that I employed for the free-field stress-energy tensor for consistency (see the section titled Homothetic Curvature). The origin of the zero-point additional energy is analogous to the non-vanishing of the zero-point energy of a simple harmonic oscillator that is seen in quantum physics. It is an indication that the transition to the point-particle description is not entirely appropriate. Note also that, due to Einstein’s equation, the covariant derivative of the particle stress-energy tensor vanishes in the absence of the free-field.

IX. THE CONCEPT OF IRREDUCIBILITY

In four dimensions the Lagrangian density must have dimension l^{-4} . Formally the metric must be dimensionless. This immediately leads to a problem with the theory presented above as follows. The Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ has dimension L^{-4} because the A^α field is given dimension l^{-1} and each derivative contributes an l^{-1} .

The contracted curvature tensor (whether homothetic or rotation), when derived from a dimensionless metric, thus has dimension l^{-2} . It is this fact that makes the coupling constant of the gravitational field l^{-2} and renders quantum gravity non-renormalisable.

Thus it appears that in performing the translation between symmetric and anti-symmetric representations of the electro-magnetic field we must also introduce a dimensional transmutation in order to give the free-field Lagrangian the appropriate dimension.

Ultimately this is the crux of the problem of unifying electro-magnetism and gravitation and also the central issue causing the difficulty quantising gravitation. Very much in the spirit of H Weyl’s ideas, I want now to explore a possible solution to this problem that centres about the issue of scale-transformations. The following is a sketch, not entirely rigorous, of the central ideas involved.

Einstein hints at the problem in his last published paper [7] when he discusses the obvious difference between the inherent discontinuity of quantum objects and the continuum of space-time; an apparent schizophrenia that has no deeper physical explanation in current theory.

Let us firstly assume that discontinuity is the fundamental element of physical structure and that the continuum is built up from a more fundamental element of structure that is ultimately completely discontinuous. This would imply that both matter and space-time are built from the same basic ‘stuff’. (A strong empirical hint that this must be the case is seen with phenomena such as creation of particle pairs from the vacuum in HEP). The most basic element of structure that could be postulated seems to me to be something like an ‘on-off’ or (0,1) duality. A plausible associated metric would be;

$$d(a, b) = |\epsilon(a, b)| \quad (48)$$

The meaning of eq.(48) is that the distance between points labelled a and b is the absolute value unity (i.e. 1) if $a \neq b$ and zero if $a = b$. This is, of course, regardless of where the points a and b happen to be located. Indeed, according to this metric it makes no sense to talk about where the points are; only that they are separate or distinguishable. No ‘background’ space-time as such exists according to this metric; we want to *build* a four-dimensional space-time out of this metric. We postulate the following algebra for the metric;

$$|\epsilon(a, b)| \cdot |\epsilon(b, c)| = |\epsilon(a, c)| \quad (49)$$

so that the product of two objects of dimension l^1 is not l^2 but l^1 . I call such an object an *irreducible interval* and its dimensionality is also set irreducibly at unity; dimensionality is thus in some sense quantised in this scheme. The *number* one is defined as a *counting* of the existence of the interval from one end to the other. Iterated countings still only define the number one. The number zero may be thought of as the non-existence of the interval or the point upon which counting is initiated.

Iterated counting may be symbolised as;

$$|\epsilon|^n = 1 \quad (\forall n \neq \aleph_0) \quad (50)$$

The object $|\epsilon|^{\aleph_0}$ with transfinite (completed infinite) index is not definable in a singular irreducible dimension. We assume it defines a two dimensional space bounded by irreducible intervals. Such a space must contain at least three points on its boundary. Its associated metric is written as;

$$\begin{aligned} d^2(a, b, c) &= |\epsilon^2(a, b, c)| = |\epsilon^2| \\ |\epsilon|^{\aleph_0} &= |\epsilon^2| \end{aligned} \quad (51)$$

The ‘area’ bounded by the irreducible intervals and defined by metric (51) I will call an ‘irreducible area’ or I.A. Its cardinality is that of the counting numbers \aleph_0 (i.e. the field of rational numbers) *not* that of the continuum. (By contrast the cardinality of the irreducible interval is strictly finite). It is this kind of object that I want to assume forms the superstructure of the photon. On the light cone we assume it doesn’t define a space with the property of the real continuum. To get a continuum we must assume the continuum hypothesis (i.e. that the next highest transfinite cardinal above \aleph_0 is c the cardinality of the continuum), and that propagation of the photon with respect to all and any observers generates such an equivalent space. The metric may be written;

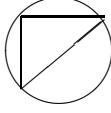
$$\begin{aligned} d^3(a, b, c, d) &= |\epsilon^3(a, b, c, d)| = |\epsilon^3| \\ |\epsilon^2|^{\aleph_0} &= |\epsilon^3| \end{aligned} \quad (52)$$

(The last equation is analogous to the equation $2^{\aleph_0} = c$). Metric (52) describes *irreducible volumes* (I.V.’s) the contained space of which is assumed to have the mathematical property of the continuum but no contained fourth dimension (no time). Such a space provides a candidate for both quantum objects with mass and propagating photons; of course for the latter we must add time as the dynamical factor generating the volume if we assume that photons are propagating I.A.’s. with respect to objects with mass. On the boundary of massive objects we will expect to find I.A.’s and thus an associated massless field.

Of course with this kind of scenario the time dimension itself is not really geometrically defined; it is an assumed added parameter. It is possible to extend the geometric/mathematical analogy to postulate a more geometric origin for time but here we will assume that the addition of time does not alter the cardinality; space-time has the same cardinality as 3-space which is that of the continuum.

Note that, even though a timeless 3-space defined by metric (52) is a continuum it is *irreducible* in the sense that it cannot be subdivided because to do so would violate the irreducibility of the bounding intervals (or equivalently the bounding areas) upon which the hierarchy of structure is built; it is in this sense quantised irreducibly and immortal. Dynamics can only occur on the boundary of the object; never in its interior.

It is possible to postulate that the physical manifestation of metrics (48), (51) and (52) is local gauge invariance. To see how this idea works geometrically consider three points selected at random on a circle consisting of a real continuum of points;



Now, we know from the work of G. Cantor that for a continuum of points a 1:1 mapping can be defined from the points on any finite length line segment onto any other line segment of arbitrary length. Thus for the continuum of points on the circle we can define a 1:1 and onto mapping of the circle onto itself which does not leave the triangle invariant; the three points defining the triangle can be shifted around the circle by such a transformation. This is another way of saying that the continuum can be compressed or stretched to an arbitrary degree without the structure of the continuum itself varying. Such a mapping is an exact analogue of a local U1 gauge transformation on the circle. However, under metric (48), and indeed only under metric (48), the triangle itself may be regarded as invariant since the angles subtended by the sides of the triangle are not defined under such a metric since each edge of the triangle always has unit length. Such a triangle is ‘irreducible’ under a local gauge transformation. Alternatively we may view the concept of the combination of an irreducible geometry embedded in the structure of the continuum as a deeper explanation of the origin of local gauge invariance itself; i.e. that the embedding of absolute discontinuity into the continuum gives rise to local gauge invariance. I have in mind here the basic foundation of quantum objects embedded in the continuum of space-time; or, alternatively in the language of the geometry presented above, of quantum objects actually *generating* the space-time continuum. Such an embedding is a fundamental union of the discrete and the continuous.

We now postulate that a photon literally has intrinsic geometric structure built up from irreducible intervals which have geometric and physical definition only on the light cone itself (a triangular geometry, for example, might be candidate) or more particularly as some form of irreducible geometry defined in a two-dimensional plane orthogonal to the direction of motion of the photon and propagating at the speed of light. The geometric irreducibility, which is inherently non-local, itself is unobservable; we see its physical manifestation *indirectly* through the unobservability of local gauge; i.e. local gauge invariance of electro-magnetism. (Of course the same must apply to the *boundary* of a three-dimensional object defined by metric (52); such a geometry is assumed to be a massive fermion quantum object and the boundary its associated electro-magnetic and gravitational fields; there must be implications here for the theory of neutrinos but I will not discuss this issue in this paper).

We can now reinterpret the translation process for the free-field electro-magnetic stress-energy tensor as follows. The anti-symmetric part of metric (12) is an irreducible metric on the light cone; this means that it does not hold in any observer’s frame. Each $\sigma_{\alpha\beta}$ term, which ultimately will contribute one A_α or A_β term, is assumed to have dimension l^{-2} and, in addition to a dimensionless phase factor $e^{\pm ik \cdot x}$, contains an intrinsic product of an I.A. to make the whole object (c.f. eq(11))

$$g_{\alpha\beta}^A = |\epsilon^2(\alpha, \beta)| \cdot e^{\pm ik \cdot x} \cdot \sigma_{\alpha\beta} = |P| \sigma_{\alpha\beta} \equiv \sigma_{\alpha\beta} \quad (53)$$

dimensionless. (The I.A. here is rather like a dimensional polarisation tensor; because of the peculiar algebra of these metrics we can still use the g_A to raise and lower indices).

Now the dimension $|\epsilon|$ is ‘infinitely smaller’ than the dimension $|\epsilon^2|$. In anticipation of imposing a scale on the irreducible metric as a part of the translation procedure let us define the irreducible area viz a term $d|\epsilon^2|$;

$$|\epsilon^2| = \int_{-\infty}^{+\infty} |\epsilon| \cdot d|\epsilon^2|$$

where $d|\epsilon^2|$ is the (infinitesimal but denumerable) increment in area in a direction orthogonal to $|\epsilon|$. The integration is carried over all space. Also we have;

$$|\epsilon^3| = \int_{-\infty}^{+\infty} |\epsilon^2| \cdot d|\epsilon^3|$$

With translation to an observer frame the I.A. ceases to exist (we must assume that it becomes absorbed into the structure of the continuum [31]) and the continuum has its dimension boosted from two to 3 + 1 dimensions. The idea

is to regard the metric $|\epsilon|$ as related to the graviton (this metric must be spin-2 since the generator of the discrete associated group S_2 , the permutation group of two objects, will not change sign with a rotation by π), the metric $|\epsilon^2|$ as related to the photon (the three-point discrete group $C3v$ of the triangle changes sign with rotation by π of one of its generator axes) and the metric $|\epsilon^3|$ as the metric of a massive spinor (viz-a-viz the 4-point discrete spinor group Td). The ‘smallness’ of $|\epsilon|$ may then be expressed in translation to the observer frame viz a Taylor expansion;

$$|\epsilon^2| \approx (\partial_\mu |\epsilon^2|) \cdot \kappa \approx \kappa^2 \quad (54)$$

Which implies that in translation to the observer frame the metric $|\epsilon|$ and the $d|\epsilon^2|$ are ‘small’ in relation to the electro-magnetic irreducible metric to the order of the gravitational constant. Notice that with translation (54) the metric equation $|\epsilon| \cdot |\epsilon| = |\epsilon|$ no longer holds because a scale has been set.

Similarly, we assume that the volume increment $d|\epsilon^3|$ is ‘small’ with respect to the volume metric $|\epsilon^3|$ to the order of the fine structure constant in relation to the mass-scale m_o of the object defined by the three-dimensional metric. Thus with translation we set;

$$|\epsilon^3| \approx |\epsilon^2| \cdot \frac{\alpha}{m_o} \approx \frac{\kappa^2 \alpha}{m_o} \quad (55)$$

Lastly we must impose an analogous set of conditions on the symmetric part of the metric;

$$g_{\alpha\beta}^S = |\epsilon_i^3| e^{ip_i \cdot x} \eta_{\alpha\beta} \quad (56)$$

where $\eta_{\alpha\beta}$ now contains a product of two fields each of dimension $l^{-\frac{3}{2}}$ i.e. spinor fields, and the irreducible volume metric of dimension l^3 appears. The i here labels particle types. We must assume that with a translation procedure from a symmetric to an anti-symmetric representation (in some sense counter-balancing the translation of the boson field A_μ from an anti-symmetric to a symmetric representation) the η field translates into a spinorial representation (using (55)) viz the ansatz;

$$|\epsilon_i^3| e^{ip_i \cdot x} \eta_{\alpha\beta} \equiv \frac{\kappa^2 \alpha}{m_o} \overline{\Psi}_i \Psi_i \gamma_\alpha \gamma_\beta \quad (57)$$

of the spinor Ψ_i . Note that the R.H.S. of eq.(57) is antisymmetric in α and β so this involves a translation between symmetric and anti-symmetric representations (i.e. we don’t equate both sides to zero!). Whether or not the Dirac Lagrangian can be extracted from this form of translation remains to be seen. The special algebra of these irreducible metrics, as before, allows the use of the total metric to raise and lower indices. The irreducible metric $|\epsilon_i^3|$ behaves algebraically like a dimensionless quantity prior to translation. Our Lagrangian reads;

$$\mathcal{L} = \kappa^{-2} \sqrt{|g|} R \quad (58)$$

X. DISCUSSION; SUPERSYMMETRY OR SUPERSLIMMETRY?

In this paper only the photon has been given ‘dual’ representations both as spinor and vector. (Equation (57) is just a speculation for further work). The ‘operator’ which interconverts the photon representations is the procedure that converts homothetic curvature due to torsion into rotation curvature. What does this mean geometrically?

Consider again the triangle embedded in the circle pictured previously. Torsion breaks parallelograms (or equivalently triangles) but under the irreducible metric the triangle does not break when subjected to torsion. In fact it’s ‘unbreakability’ under the torsion induced homothetic curvature is nothing other than an expression of U1 local-gauge invariance as was previously demonstrated. But this is a compact rotation symmetry! Thus we see that the inter-conversion of bosonic and fermionic representations is bridging compact and non-compact groups because the torsion is the generator of translations. This enables us to have a more fundamental physical reason for the occurrence of local gauge invariance in nature; invariance of the structure of the continuum to arbitrary deformations. It is the structure of the continuum which is truly fundamental; the matter fields and the forces between them appear as the superstructure keeping the continuum continuous.

Looking at the invariance of the irreducible geometry used to describe the photon under torsion induced translation is equivalent, at least from the geometric point of view, to ‘dressing’ the photon with its own gravitational self-interaction. To see this note that, since torsion breaks parallelograms if the triangle were defined by ordinary geometry it would break if the intervals defining it were not irreducible. Consider the simplest break; a rupture at one of the

vertices of the triangle. The result will be an object with four vertices. The extra interval, under the assumptions presented, is the geometry of a graviton. The restoration of the geometry of the triangle would then correspond to the resorption of the emitted graviton. In this manner irreducibility of the geometry of the triangle - that is, its invariance under torsion induced translations - is equivalent to ‘dressing’ the photon with the gauge field of translations; its own gravitational self-interaction. Eq.(54) gives us a scale for the interaction; each photon is dressed by gravitons of the order of the Planck length. Thus the ‘breaks’ induced by torsion are extremely small scale.

A similar interpretation can be given to eq.(55); irreducibility equals local gauge invariance and here the global $U(1)$ phase of the photon is the gauge group restoring invariance of the phase of the source of the electro-magnetic field. The irreducible metric is implicitly including the gravitational and electromagnetic self-interactions of the the source; the source is ‘dressed’ by its own fields.

It seems to me that we have the following senario. We have dual descriptions of electromagnetism. Interpreted in an observer frame torsion and its induced homothetic curvature is unobservable; it could only ever appear as a non-propagating contact interaction. However, to an observer ‘travelling on a ray of light’ (which, for the sake of explanatory convenience, we shall admit) the torsion and homothetic curvature would appear real and the observers world would be a strange place where photons behave as spinors and only the anti-symmetric part of the photon’s stress-energy tensor is a conserved quantity. To our observer riding on a photon the photon obeys a Pauli exclusion principle; which is to say in a space where $ds^2 = 0$ our observer is in no position to ‘see’ any other photon other than the one he or she is unfortunate enough to be ensconced with.

Then there is the other description of electromagnetism with which we are more familiar. In it the photons are bosons and the conserved stress-energy tensor is of course symmetric. In this frame the homothetic curvature to our observer perched on a photon appears to the more familiar observer on Earth as rotation of space-time in the local vicinity of each individual photon when we call it the wave nature of light.

It is in this manner that the C-M theorem may be overcome; not by supersymmetry but with a slimmer menagerie of fundamental objects - which is of course desirable - with each particle providing its own ‘superpartner’ of which the photon, in this case, may provide the prototype example. This seems to me more natural than conventional supersymmetry given that it generates local gauge invariance rather than assuming it and does not generate unobserved objects.

Thus unification in four dimensions is not yet a closed subject and hopefully this paper has stimulated some interest in it. In particular obtaining a gauge-invariant traceless stress-energy tensor for the free-field is quite a non-trivial result peculiar to the mathematical structure I have presented. Note that exactly the right components must be present in the expansion of the curvature tensor for the mathematics to work. Variation of the Lagrangian (58) now leads to the Einstein equation on the L.H.S. and the sum of the free-field and particle stress-energy tensors on the R.H.S. and both gravitation and electro-magnetism are accommodated in the single equation. The unification of the gauge couplings is speculative but is assumed to be linked to the structure of the continuum beyond the Planck scale. Using irreducible metrics means that we really must go beyond the conventional conception of space-time. Instead of a fifth dimension to define electric charge, particles now appear rather like non-local bubbles in the vacuum inside which time is absent. The closer we look at the bubble the smaller it gets (I have in mind here electrons and quarks). The quantisation of charge must now be related to the topology of the boundary of this space. The decomposition of the vacuum is more severe on the light cone where the structure of the continuum itself is actually altered.

It would be appropriate to summarise what has been done in order to get the mathematical content in perspective. Firstly the metric structure of space-time has been generalised;

1. to include a $U(1)$ phase factor ‘on the light-cone’ the generator of which is the photon momentum. The sigma matrices in some sense ‘carry’ the representation $e^{\pm ik \cdot x}$ on the light cone. The phase factor itself means that, in the presence of the photon, space on the light cone has an intrinsic wave-structure. The associated torsion is generating, viz the homothetic curvature, the corresponding free-field stress-energy tensor. This, however, is not really a true Riemann-Cartan geometry because the torsion on the light cone translates to non-torsional rotation curvature in the ‘observer frame’. The ‘frame’ in which this torsion is defined is ‘on the light cone’ i.e.; it is not an observer frame. To reposition the representation in an observer frame it must be translated from an anti-symmetric representation into a symmetric representation. This is analogous to transforming homothetic curvature due to torsion into rotation curvature. Once translated the stress-energy tensor for the free-field must be added to the rotation curvature coming from the source as both must now be regarded as contributing to the rotation curvature.

2. The second generalisation involves adding a $U(1)$ phase factor related to the charged source momentum with non-zero rest mass to the symmetric part of the metric. This is ‘on the observer frame’ (i.e. $ds^2 \neq 0$) and generates rotation curvature in a manner analogous to the rotation curvature generated in Gravitation theory. The potential generating a current in this case is then the 4-momentum of the charged electron or proton. This current is a conserved quantity;

$$-\frac{i}{4}\Gamma_{\gamma\alpha}^{\alpha} = (-i\partial_{\gamma}e^{ip\cdot x})e^{-ip\cdot x} = p_{\gamma} = j_{\gamma} \quad (59)$$

so that clearly $\partial^{\gamma}j_{\gamma} = 0$ when we treat momentum and position as independent variables in the quantum representation. The presence of a phase factor in the metric for an electron means that

$$p^2 = m_o^2 e^{ip\cdot x}$$

but if we reinterpret this in terms of operators and wave-functions we have instead;

$$\hat{p}^{\mu}\hat{p}^{\nu}g_{\mu\nu} = \hat{p}^{\mu}\hat{p}^{\nu}\eta_{\mu\nu} < x | p > = m_o^2 < x | p >$$

Thus at short distance scales (there is an intrinsic factor of \hbar in the phase factor) space-time takes on a wave-nature which means that a classical particle description is no longer appropriate. In particular the non-zero zero point energy present in the stress-energy tensor means that we are dealing with an harmonic oscillator with non-zero minimum energy. In the framework of general relativity the wave part of the wave-particle duality is thus due to space-time curvature at short scales; it's another way of looking at the world. Unfortunately the precise quantitative difference in the scales of the coupling parameters for electro-magnetism and gravitation has not been given an explanation but it has been noted that, prior to translation, the difference is infinite! It is thus perhaps no surprise that there results with translation a large difference (at least at low energies).

After translating the free-field stress-energy tensor to a symmetric form we may consider it a component part of the rotation curvature generated by metric (40) instead of homothetic curvature. With this proviso summing the contributions to rotation curvature arising from the metric (40) we may write;

$$\begin{aligned} -\frac{1}{\kappa^2}(R^{\gamma\delta} - \frac{1}{2}\eta^{\delta\gamma}R) &= \frac{\alpha}{2}m_o U^{\gamma}U^{\delta}\frac{d\tau}{dx^o}\delta^3(x^i - x_p^i(\tau)) \\ + F^{\gamma\alpha}F_{\alpha}^{\delta} + \frac{1}{4}\eta^{\gamma\delta}F^{\mu\nu}F_{\mu\nu} &= T_P^{\gamma\delta} + T_F^{\gamma\delta} \end{aligned} \quad (60)$$

where the zero-point energy has been discarded as is usually done in quantum theory. The covariant derivative of both sides of eq.(60) must vanish which gives us the inhomogenous Maxwell equations. The homogenous Maxwell equations result from considering the free-field alone (i.e. equation (35)) by setting source 4-momentum to zero. The classical equations of motion for a point particle in an electro-magnetic field result if we substitute the ordinary derivative for the covariant derivative. Thus we can recover classical electro-dynamics. Also note that we can also add to the right side of eq.(60) the contributions from gravitation by letting the symmetric part of the metric vary with the gravitational potentials. It will of course enter with the opposite sign to the contributions from the electromagnetic field. For macroscopic charged matter we would of course have to integrate up the source term in eq.(60). An object with opposite charge will, however, enter with opposite sign if we set $\eta_{\alpha\beta}e^{ip\cdot x} = \text{unit negative charge}$ and $\eta_{\alpha\beta}e^{-ip\cdot x}$ as unit positive charge say.

In forming a combined gravitational and electromagnetic curvature equation it must however be remembered that the curvature is no longer purely gravitational in nature. At the micro-scale of space-time there is severe curvature due to mass carrying electric charge which is not gravitational in nature.

Does Einstein's metric theory of Gravity remain intact under the above derivation of the electro-magnetic stress-energy tensor? Does the equivalence principle still hold?

To answer these questions requires some interpretation of the above derivation and metric (12). Noting that the derivation of the free-field electro-magnetic stress-energy tensor was carried out with a completely antisymmetric metric and that;

$$g_{\alpha\beta}^A dx^{\alpha}dx^{\beta} = ds^2 = 0$$

we can interpret the antisymmetric part of metric (12) as a 'co-moving' metric in the light-cone frame of the photon; the null geodesic. As noted above, one consequence of this is that Lorentz scalars for macroscopic matter remain invariant under the total metric (i.e. the combination of symmetric and anti-symmetric parts) and we avoid the sort of problems related to measuring rods and clocks that led to so much criticism of the Weyl/Eddington theory. (I believe that at one stage Einstein himself attempted to construct a version of the Weyl/Eddington theory 'on the light cone' to avoid the associated measurement problems [5]). The torsion I interpret as an essentially 'local' phenomena in the vicinity of each individual photon. (See [8] [9] for similar ideas).

For $g = g_a + g_s$ the vanishing of the covariant derivative of the metric employed in the derivation of the connection means that the (strong) equivalence principle applies to the symmetric part of the metric which leads to gravitation theory. The additional presence of a phase factor at the quantum scale will be expected to vanish for bulk matter as might be expected in the classical limit. Thus the classical theory of general relativity remains intact.

Notice that (using eq (19) and the definition of A^μ);

$$\sigma^{\alpha\beta}_{,\alpha} = -i\sigma^{\alpha\beta}_{,\gamma}\sigma^\gamma_\alpha = +i\sigma^\gamma_\alpha\sigma^{\alpha\beta}_{,\gamma} = -\sigma^{\alpha\beta}_{,\alpha}$$

so that $\partial_\mu A^\mu = 0$ and the connection imposes the Lorentz condition. This is the constraint in Proca's equation which allows torsion for $m \neq 0$ [8] which the co-moving metric makes implicit at $m = 0$.

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